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A Class of Einstein-Maxwell Fields Generalizing the Equilibrium Solutions

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Abstract

The Einstein-Maxwell fields of rotating stationary sources are represented by the SU(2,1) spinor potential Ψ_A satisfying

$$\nabla \cdot [\Theta^{-1}(\Psi_A \nabla \Psi_B - \Psi_B \nabla \Psi_A)] = -2\Theta^{-2}\vec{C} \cdot (\Psi_A \nabla \Psi_B - \Psi_B \nabla \Psi_A)$$

where $\Theta = \Psi^\dagger \cdot \Psi$ is the $SU(2,1)$ norm of Ψ. The Ernst potentials are expressed in terms of the spinor potential by $\mathcal{E} = \frac{\Psi_1 - \Psi_2}{\Psi_1 + \Psi_2}$, $\Phi = \frac{\Psi_3}{\Psi_1 + \Psi_2}$. The group invariant vector $\vec{C} = -2i\,\mathrm{Im}\{\Psi^\dagger \cdot \nabla \Psi\}$ is generated exclusively by the rotation of the source, hence it is appropriate to refer to $\vec{C}$ as the *swirl* of the field. Static fields have no swirl.

The fields with no swirl are a class generalizing the equilibrium ($|e| = m$) class of Einstein-Maxwell fields. We obtain the integrability conditions and a highly symmetrical set of field equations for this class, as well as exact solutions and an open research problem.

I. Introduction: Einstein-Maxwell fields of rotating stationary sources

A stationary Einstein-Maxwell field is characterized by the two complex Ernst potentials[1] $\mathcal{E}$ and Φ of the gravitational and electromagnetic fields, respectively. For the investigation of the internal symmetries of the system, it is more advantageous to use a pair of new potentials ξ and q, defined by the relations

$$\mathcal{E} = \frac{\xi - 1}{\xi + 1} \qquad \Phi = \frac{q}{\xi + 1}. \tag{1}$$

In terms of the ξ and q potentials, the Lagrangian of the Einstein-Maxwell system takes the form

$$L = \theta^{-2}[\partial_\mu \xi \partial^\mu \bar{\xi} - \partial_\mu q \partial^\mu \bar{q} + (\xi \partial_\mu q - q \partial_\mu \xi)(\bar{\xi} \partial^\mu \bar{q} - \bar{q} \partial^\mu \bar{\xi})] \tag{2}$$

with

$$\theta = \xi \bar{\xi} + q \bar{q} - 1.$$

Here the metric $g_{\mu\nu}$ is that of the Euclidean 3-space in any suitable coordinate system. Hence the Euler-Lagrange equations follow:

$$\begin{aligned} \theta \Delta \xi - 2(\bar{\xi} \nabla \xi + \bar{q} \nabla q) \cdot \nabla \xi &= 0 \\ \theta \Delta q - 2(\bar{\xi} \nabla \xi + \bar{q} \nabla q) \cdot \nabla q &= 0. \end{aligned} \tag{3}$$

The global symmetries of this system form the $SU(2,1)$ group.[3] In the next section, we shall briefly review the basic theory of conserved currents and their relation to global symmetries. We then compute the conserved currents of the Einstein-Maxwell system and find a highly symmetrical form of the field equations using the currents. We introduce the concept of the 'swirl vector' $\vec{C}$ which is a group invariant. In the subsequent sections the fields satisfying the condition of vanishing swirl are investigated.

II. Symmetries and Currents

The action as a functional of the fields ϕ^i and $\partial_\mu \phi^i$ is

$$S = \int d^n x \mathcal{L}(\phi^i, \partial_\mu \phi^i).$$

Associated with a symmetry transformation

$$\phi^i \to \phi^i + \delta \phi^i ,$$

such that the Lagrangian density is invariant, $\delta \mathcal{L} = 0$, and

$$\delta \phi^i = \varepsilon^i_k \phi^k, \tag{4}$$

there exists a conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^i\right)} \varepsilon^i_k \phi^k \tag{5}$$

(*Noether* current) satisfying

$$\partial_\mu J^\mu = 0 \ .$$

We now proceed with the application of this general theory to the Einstein-Maxwell system. The Lagrangian (2) possesses an $SU(2,1)$ global symmetry group[2].[3] The potentials α, β and γ belonging to the fundamental representation of this group are given by

$$\xi = \frac{\alpha}{\beta} \qquad\qquad q = \frac{\gamma}{\beta}. \tag{6}$$

The global invariance transformations are

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \rightarrow \mathbf{U} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{7}$$

where $\mathbf{U} \in SU(2,1)$ is a constant matrix.

The defining representation of the global symmetry is given by the spinor potential $\Psi_A = (\alpha, \beta, \gamma)$, and its group adjoint $\Psi^{\dagger A} = (\bar{\alpha}, -\bar{\beta}, \bar{\gamma})$. Hence also the Ernst potentials are written

$$\mathcal{E} = \frac{\Psi_1 - \Psi_2}{\Psi_1 + \Psi_2} \qquad\qquad \Phi = \frac{\Psi_3}{\Psi_1 + \Psi_2}.$$

These spinors are determined up to an overall complex multiplying function; the equivalence classes are given by the relation

$$\Psi'_A = \Omega \Psi_A \tag{8}$$

where Ω is an arbitrary complex scalar.

The field equations (3) will take the $SU(2,1)$ invariant form

$$\nabla \cdot [\Theta^{-1}(\Psi_A \nabla \Psi_B - \Psi_B \nabla \Psi_A)] = -2\Theta^{-2} \vec{C} \cdot (\Psi_A \nabla \Psi_B - \Psi_B \nabla \Psi_A) \tag{9}$$

where $\Theta = \Psi^\dagger \cdot \Psi$ is the $SU(2,1)$ norm of Ψ.

We now turn our attention to the group invariant vector

$$\vec{C} = -2i \, \mathrm{Im}\{\Psi^\dagger \cdot \nabla \Psi\}.$$

For static fields, all potentials are real, hence we have $\vec{C} = 0$. We thus infer that the vector $\vec{C}$ is a concomittant of the rotation of the source, so that we will call $\vec{C}$ the *swirl* of the field. The form of the vector $\vec{C}$ depends on the gauge (8). In this paper, we consider Einstein-Maxwell fields for which it is possible to choose a gauge in which the swirl vanishes:

$$\vec{C} = 0. \tag{10}$$

In addition to static metrics, this condition characterizes the equilibrium ($|e| = m$) class of fields.

The currents (5) of the SU(2,1) symmetry can be expressed by use of the swirl $\vec{C}$ as follows:

$$\begin{aligned}
J^{(1)} &= \Theta^{-2}(\alpha\bar{\alpha} + \beta\bar{\beta})\vec{C} + \Theta^{-1}(\bar{\alpha}\nabla\alpha - \alpha\nabla\bar{\alpha} + \ \bar{\beta}\nabla\beta - \beta\nabla\bar{\beta}) \\
J^{(2)} &= \Theta^{-2}\alpha\bar{\beta}\vec{C} + \Theta^{-1}(\bar{\beta}\nabla\alpha - \alpha\nabla\bar{\beta}) \\
J^{(3)} &= \Theta^{-2}\beta\bar{\alpha}\vec{C} + \Theta^{-1}(\bar{\alpha}\nabla\beta - \beta\nabla\bar{\alpha}) \\
J^{(4)} &= \Theta^{-2}\alpha\bar{\gamma}\vec{C} + \Theta^{-1}(\bar{\gamma}\nabla\alpha - \alpha\nabla\bar{\gamma}) \\
J^{(5)} &= \Theta^{-2}\gamma\bar{\alpha}\vec{C} + \Theta^{-1}(\bar{\alpha}\nabla\gamma - \gamma\nabla\bar{\alpha}) \\
J^{(6)} &= \Theta^{-2}\beta\bar{\gamma}\vec{C} + \Theta^{-1}(\bar{\gamma}\nabla\beta - \beta\nabla\bar{\gamma}) \\
J^{(7)} &= \Theta^{-2}\gamma\bar{\beta}\vec{C} + \Theta^{-1}(\bar{\beta}\nabla\gamma - \gamma\nabla\bar{\beta}) \\
J^{(8)} &= \Theta^{-2}\gamma\bar{\gamma}\vec{C} + \Theta^{-1}(\bar{\gamma}\nabla\gamma - \gamma\nabla\bar{\gamma}).
\end{aligned} \tag{11}$$

Here

$$\Theta = \alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma}$$

and the swirl has the detailed form

$$\vec{C} = \alpha\nabla\bar{\alpha} - \beta\nabla\bar{\beta} + \gamma\nabla\ \bar{\gamma} - \bar{\alpha}\nabla\alpha + \bar{\beta}\nabla\beta - \bar{\gamma}\nabla\gamma.$$

From $J^{(1)}$ and $J^{(8)}$ we get

$$\begin{aligned}
J^{(1a)} &= \Theta^{-2}\alpha\bar{\alpha}\vec{C} - \Theta^{-1}(\alpha\nabla\bar{\alpha} - \bar{\alpha}\nabla\alpha) \\
J^{(1b)} &= \Theta^{-2}\beta\bar{\beta}\vec{C} - \Theta^{-1}(\beta\nabla\bar{\beta} - \bar{\beta}\nabla\beta).
\end{aligned}$$

For fields satisfying the condition of vanishing swirl, $\vec{C} = 0$, the Einstein-Maxwell Eqs. (9) can be written in the simple form

$$\nabla \cdot [\Theta^{-1}(\Psi_A\nabla\Psi_B - \Psi_B\nabla\Psi_A)] = 0 \tag{12}$$

From the equations (11) of current conservation we get the symmetrical set

$$\nabla \cdot [\Theta^{-1}(\Psi^{\dagger A}\nabla\Psi_B - \Psi_B\nabla\Psi^{\dagger A})] = 0. \tag{13}$$

Equations (12) and (13) govern the class of Einstein-Maxwell fields characterized by a vanishing swirl.

III. Formulation using the vectors G and H:

An Einstein-Maxwell system with one Killing vector may be fully characterized by the complex 3-vectors[4]:

$$\mathbf{G} = \frac{\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi}{2(\mathrm{Re}\mathcal{E} + \bar{\Phi}\Phi)}, \qquad \mathbf{H} = \frac{\nabla\Phi}{(\mathrm{Re}\mathcal{E} + \bar{\Phi}\Phi)^{1/2}}.$$

In the notation referring to the metric of the three-space, the field equations can be written

$$R_{\mu\nu} = -G_\mu\bar{G}_\nu - \bar{G}_\mu G_\nu + H_\mu\bar{H}_\nu + \bar{H}_\mu H_\nu \tag{14}$$

$$\begin{aligned}
(\nabla - \mathbf{G})\cdot\mathbf{G} &= \bar{\mathbf{H}}\cdot\mathbf{H} - \bar{\mathbf{G}}\cdot\mathbf{G} && (15)\\
(\nabla - \mathbf{G})\times\mathbf{G} &= \bar{\mathbf{H}}\times\mathbf{H} - \bar{\mathbf{G}}\times\mathbf{G} && (16)\\
(\nabla - \mathbf{G})\cdot\mathbf{H} &= \frac{1}{2}(\mathbf{G} - \bar{\mathbf{G}})\cdot\mathbf{H} && (17)\\
\nabla\times\mathbf{H} &= -\frac{1}{2}(\mathbf{G} + \bar{\mathbf{G}})\times\mathbf{H}. && (18)
\end{aligned}$$

The vectors $\mathbf{G}$ and $\mathbf{H}$ can be expressed in terms of the gradients of the complex potentials as follows,

$$\begin{aligned}
\Theta\mathbf{G} &= \left(\bar{\xi} + 1 - q\bar{q}\right)\frac{\nabla\xi}{\xi + 1} + \bar{q}\nabla q && (19)\\
\Theta^{1/2}\mathbf{H} &= \left(\frac{\bar{\xi} + 1}{\xi + 1}\right)^{1/2}\left(\nabla q - q\frac{\nabla\xi}{\xi + 1}\right). && (20)
\end{aligned}$$

Solving for the gradients, we have

$$\begin{aligned}
\nabla\xi &= \Theta\frac{\xi + 1}{\bar{\xi} + 1}\mathbf{G} - \Theta^{1/2}\bar{q}\left(\frac{\xi + 1}{\bar{\xi} + 1}\right)^{3/2}\mathbf{H} && (21)\\
\nabla q &= \left(\bar{\xi} + 1\right)^{-1} q\Theta\mathbf{G} + \left(\bar{\xi} + 1 - q\bar{q}\right)\Theta^{1/2} q\left(\xi + 1\right)^{1/2}\left(\bar{\xi} + 1\right)^{-3/2}\mathbf{H}.
\end{aligned}$$

By use of the definitions (6) and the vanishing of the vector $\vec{C}$, we get the relation

$$2i\mathrm{Im}\left(\frac{\xi\nabla\bar{\xi} + q\nabla\bar{q}}{\xi\,\bar{\xi} + q\bar{q} - 1}\right) = \nabla\left(\ln\frac{\bar{\beta}}{\beta}\right).$$

Inserting here the gradients (21) , we obtain the integrability conditions of $\bar{\beta}/\beta$ in the form:

$$\mathbf{G} \times \bar{\mathbf{G}} = \mathbf{H} \times \bar{\mathbf{H}}. \tag{22}$$

These constraints are apparently milder than the equilibrium ($R_{ij} = 0$) condition $Re(\mathbf{G} \otimes \bar{\mathbf{G}} - \mathbf{H} \otimes \bar{\mathbf{H}}) = \mathbf{0}$ characterizing the Einstein-Maxwell fields with balanced electromagnetic and gravitational forces.

With the help of the integrability condition (22), Eq. (16) takes the simple form

$$\nabla \times \mathbf{G} = 0. \tag{23}$$

Hence there exists a complex potential ψ such that

$$\mathbf{G} = \nabla \ln \psi.$$

Using this in Eq. (18),

$$\nabla \times \mathbf{H} = \frac{1}{2} \nabla(\ln \psi\bar{\psi}) \times \mathbf{H}.$$

Hence

$$\mathbf{H} = (\psi\bar{\psi})^{-1/2} \nabla\chi \tag{24}$$

for some complex function χ. Eq. (14) takes the form

$$R_{\mu\nu} = -2 \frac{\psi_{(,\mu}\bar{\psi}_{,\nu)} - \chi_{(,\mu}\overline{\chi}_{,\nu)}}{\psi\bar{\psi}}.$$

The essential field equations can now be written down in terms of the complex potentials ψ and χ as follows,

$$\nabla \cdot [\psi^{-2}(\overline{\psi}\nabla\psi - \overline{\chi}\nabla\chi)] = 0 \tag{25}$$

$$\nabla \cdot (\psi^{-2}\nabla\chi) = 0 \tag{26}$$

$$\nabla\psi \times \nabla\overline{\psi} - \nabla\chi \times \nabla\overline{\chi} = 0. \tag{27}$$

These are two complex and one real equations for the two complex functions χ and ψ. Despite this apparent overdetermination of the system, at least two large classes of solutions exist. One of these consists of stationary fields that are $SU(2,1)$ rotations of a static state. This is due to the fact that the potentials are real for static fields and that the vector $\vec{C}$ is $SU(2,1)$ invariant. The other subset of solutions of these equations has the form $\chi = \psi$. Field Eqs. (25)-(27) then reduce to the simple condition that ψ^{-1} is a harmonic function. This is the equilibrium class characterized by a vanishing Ricci tensor.

IV. Nonstatic fields

An example of the $SU(2,1)$ global transformations generating a nonstatic state from a given static electrovacuum is the Kramer-Neugebauer transformation,[6].[7] This has the form

$$\alpha' = (1 - z\bar{z})\alpha, \qquad \beta' = (1 + z\bar{z})\beta - 2\bar{z}\gamma, \qquad \gamma' = \left(\frac{\bar{z}}{z}\right)^{1/2} [(1 + z\bar{z})\gamma - 2z\beta] \tag{28}$$

where z is a complex constant and (α, β, γ) is the real triplet of potentials for a static solution. Choosing (α, β, γ) to be the potentials of an asymptotically flat electrovacuum, the metric so obtained remains asymptotically flat.

There is a wide range of static electrovacuum space-times which is suitable to be chosen as the seed metric of global symmetry transformations. All vacuum Weyl metrics and their electrovacuum counterparts, characterized by the condition $q = const.$, belong to these. A further simple case of an explicit static electrovacuum is the *Bonnorized*[5] Kerr metric in oblate spheroidal coordinates (x, y):

$$\alpha = x^2 + p^2 - q^2y^2, \qquad \beta = 2px, \qquad \gamma = 2pqy \tag{29}$$

where $p^2 + q^2 = 1$. This solution describes the gravitational and magnetic field of a dipole source. An infinite sequence of static solutions is given by the *Bonnorized* Tomimatsu-Sato metrics.[8] A wealth of nonstationary solutions is readily available by the $SU(2,1)$ rotations of these metrics. Among these, the rotating solutions obtained by the Kramer-Neugebauer transformation (28) will possess controllable asymptotic properties.

V. Conclusions

The Einstein-Maxwell fields with no swirl comprise a large number of space-times, and among those ones with astrophysical significance. In addition to the equilibrium ($|e| = m$) electrovacua, all static fields as well as their stationary counterparts resulting from the global $SU(2,1)$ symmetry belong to this class. To be able to better assess the extent of generality of this class, it would be clearly be of much help to find the solution of the following open research problem:

Find an example which is neither an equilibrium solution nor an $SU(2,1)$ rotation of a static field.

Likewise, it would be of considerable interest to see if the Kinnersley[3] group K' has a subgroup leaving the condition of vanishing swirl invariant.

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References

[1]F.J. Ernst, Phys. Rev. **168,** 1415 (1968)

[2]G.Neugebauer and D.Kramer, Ann. Physik **24**, 62 (1969)

[3]W.Kinnersley, J.Math.Phys. **18**, 1529 (1977)

[4]Z.Perjés, Commun.Math.Phys. **12**, 275 (1969)

[5]W.Bonnor, Z.Phys. **161**, 439 (1961)

[6]D.Kramer and G.Neugebauer, Ann. Physik **24**, 59 (1969)

[7]F.P.Esposito and L.Witten, Phys.Rev. **D8**, 3302 (1973)

[8]For the Bonnorized $\delta = 2$ Tomimatsu-Sato metric, cf. Z. Perjés, in General Relativity, Eds. G.S.Hall and J.R.Pulham, IOP, 1955, p.103